Lecture 23

extra OH Fri?

Plan: 1) finish stating ellipsoid
2) Analyze ellipsoid
3) Apply to LP. ← Next time

Analysis of ellipsoid

Recall main lemma:

Volume Lemma: Let E' be ellipsoid after E in the algorithm.

Then:

$$\mathrm{vol}(E') \le e^{-\frac{1}{2(n+1)}} \mathrm{vol}(E).$$

Before proving, some preliminaries:

need A P.D, or else ellipsoid will contain an entire line.

Def: Given center $e \in \mathbb{R}^n$, & a positive definite matrix $A \in \mathbb{R}^{n\times n}$, the ellipsoid $E(e,A)$ is given by

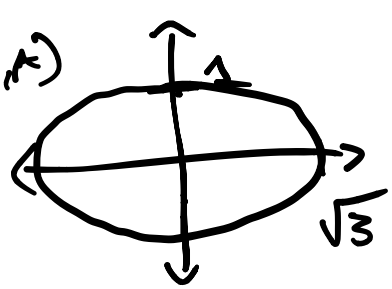
$$E(e,A) := \{x \in \mathbb{R}^n : (x-e)^T A^{-1}(x-e) \le 1\}.$$

e.g. $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad A' = R(\theta)\, A\, R(\theta)^T$

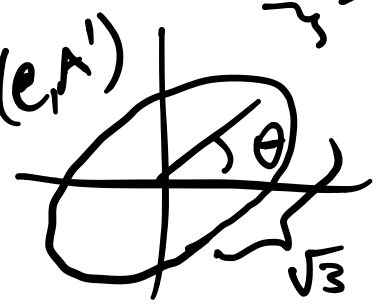
$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$e = (0,0)$

$E(e,A)$



$E(e,A')$



$A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$, $(x,y)^T A^{-1} (x,y) = \frac{x^2}{3} + y^2 \le 1$

Recall: Matrix $A \in \mathbb{R}^{n\times n}$ positive-definite

if 1) A is symmetric $A^T = A$

& 2) $x^T A x > 0 \ \forall x \in \mathbb{R}^n \setminus \{0\}$.

Equivalent conditions: Let A be a symetric matrix. then:

A P.D.

$\iff \exists B \in \mathbb{R}^{n\times n}$ s.t. $A = B^T B$

$\iff A^{-1}$ P.D. $\quad (A^{-1} = B^{-1} (B^{-1})^T)$

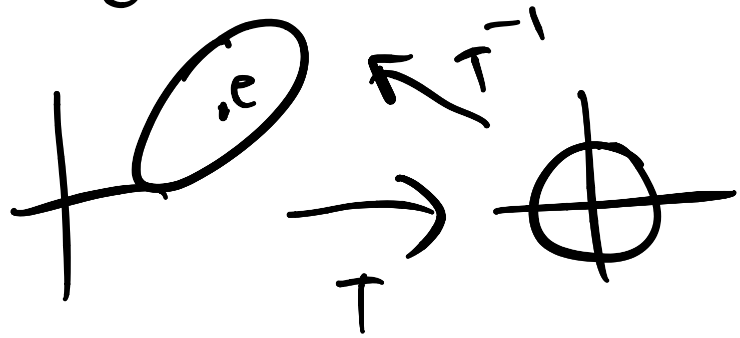
$\iff A$ has n orthonormal eigenvectors w/ positive eigenvalues.

Facts about ellipsoids:

- They are <u>affine transformations</u> (linear map + translation) of unit ~~spheres~~. balls.

$$E(0,I) = \{x \in \mathbb{R}^n : x^T x \le 1\}.$$

Proof:



Let $A = B^T B$;

$$E(0,I) = T\,E(e,A)$$

where T affine bijection

$$T : x \longmapsto y := (B^{-1})^T (x - e).$$

$y \in$ unit ball

$\Leftrightarrow y^T y \le 1 \Leftrightarrow (x-e)^T B^{-1} (B^{-1})^T (x-e) \le 1$

$$(x-e)^T A^{-1} (x-e) \le 1$$

$\Leftrightarrow x \in E(e,A)$.

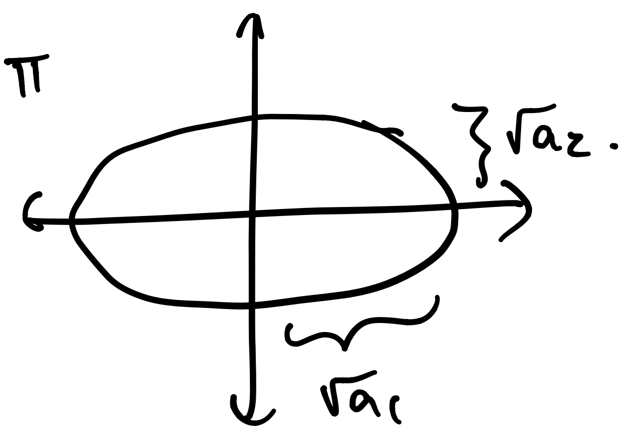
- **Volume**: if $A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$,

i.e. $E(e,A)$ "coordinate aligned"

then

$$\text{vol } E(e,A) = \sqrt{a_1 \cdots a_n} \text{ vol } E(0,I).$$

$\text{vol} = \sqrt{a_1 a_2}\,\pi$



$\det (B^{-1})^T$
$= \det B^{-1}$
$= \frac{1}{\det B}$.

$A = B^T B$

(because $\text{vol } E(e,A) = \text{vol } E(0,A)$,

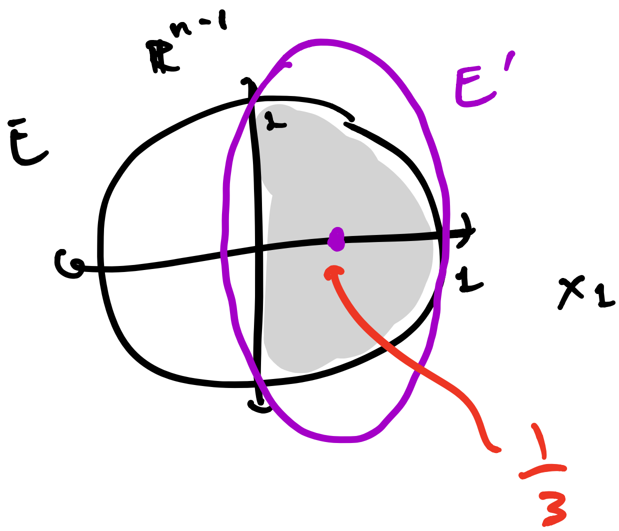
$\& \, (B^{-1})^T E(0,A) = E(0,I) \Rightarrow E(0,A) = B^T E(0,I).$

$\Rightarrow \text{vol}\, E(0,A) = |\det B| \text{vol}\, E(0,I)$

$= \sqrt{\det A}\, \text{vol}\, E(0,I) = \sqrt{a_1 \cdots a_n}\, \text{vol}\, E(0,I).$

Proof of Volume Lemma:

- Begin with special case $E = E(0,I)$ (unit sphere), & inequality $x_1 \geq 0$.



all other cases reduce to this one! by transformations.

- **Claim**: We can take

$$E' = \left\{ x : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}.$$

i.e. $E' = E(e, A)$ where $e = \left(\frac{1}{n+1}, 0, \dots, 0\right)$

and $A = \operatorname{diag}\left(\left(\frac{n}{n+1}\right)^2, \frac{n^2}{n^2-1}, \dots \frac{n^2}{n^2-1}\right)$

Proof of claim: exercise: E' is actually min. volume ellipsoid $\supseteq E$; don't need.

- Need to show $E \cap \{x : x_1 \geq 0\} \subseteq E'$.
- Let $x \in E \cap \{x : x_1 \geq 0\}$. Then

want to show ≤ 1.

$$\left(\frac{n+1}{n}\right)^2 \underbrace{\left(x_1 - \frac{1}{n+1}\right)^2}_{\text{expand}} + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2$$

$$= \underbrace{\frac{n^2+2n+1}{n^2} x_1^2}_{\text{move } \frac{n^2-1}{n^2}x_1} - \underbrace{\left(\frac{n+1}{n}\right)^2 \frac{2x_1}{n+1}}_{\text{cancel}} + \frac{1}{n^2} + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2$$

$$= \underbrace{\frac{2n+2}{n^2} x_1^2 \qquad \frac{2n+2}{n^2} x_1}_{\text{collect}} + \frac{1}{n^2} + \frac{n^2-1}{n^2} \sum_{i=1}^{n} x_i^2$$

$$= \frac{2n+2}{n^2} \underbrace{x_1(x_1-1)}_{\substack{\le 0 \text{ b/c} \\ 0 \le x_1 \le 1}} + \frac{1}{n^2} + \frac{n^2-1}{n^2} \underbrace{\sum_{i=1}^{n} x_i^2}_{\substack{\le 1 \text{ b/c} \\ \|x\| \le 1}}$$

$$\le \frac{1}{n^2} + \frac{n^2-1}{n^2} \le 1. \quad \square$$

- Proof of volume lemma in this case:

$$\underbrace{\text{vol } E(c, A)}_{\text{vol } E'} = \sqrt{a_1 \cdots a_n}\, \underbrace{\text{vol } E(0, I)}_{\text{vol } E};$$

$$\sqrt{a_1 \cdots a_n} = \sqrt{\left(\frac{n}{n+1}\right)^2 \frac{n^2}{n^2-1} \cdots \frac{n^2}{n^2-1}}$$

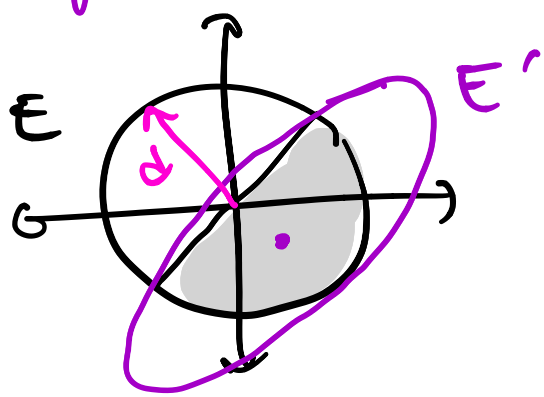
$$= \frac{n}{n+1} \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} = \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}}$$

$$\le e^{-\frac{1}{n+1}} e^{\frac{n-1}{2}\left(\frac{1}{n^2-1}\right)}$$

$$= e^{-\frac{1}{n+1}} e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}. \quad \square$$

used inequality $1+x \le e^x$

- what if we have some other inequality $d^T x \le 0$?



▷ can assume $\|d\| = 1$ by $d \leftarrow \frac{d}{\|d\|}$

▷ Figure out E' by rotating so $d = -e_1$, using previous case, then rotating back,

shows vol. ratio still $\le \exp\left(-\frac{1}{2(n+1)}\right)$.

▷ End up with $E' = E\left(-\frac{d}{n+1}, F\right)$,

$$F = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} d d^T\right)$$

d unit vector.

(check F positive definite).

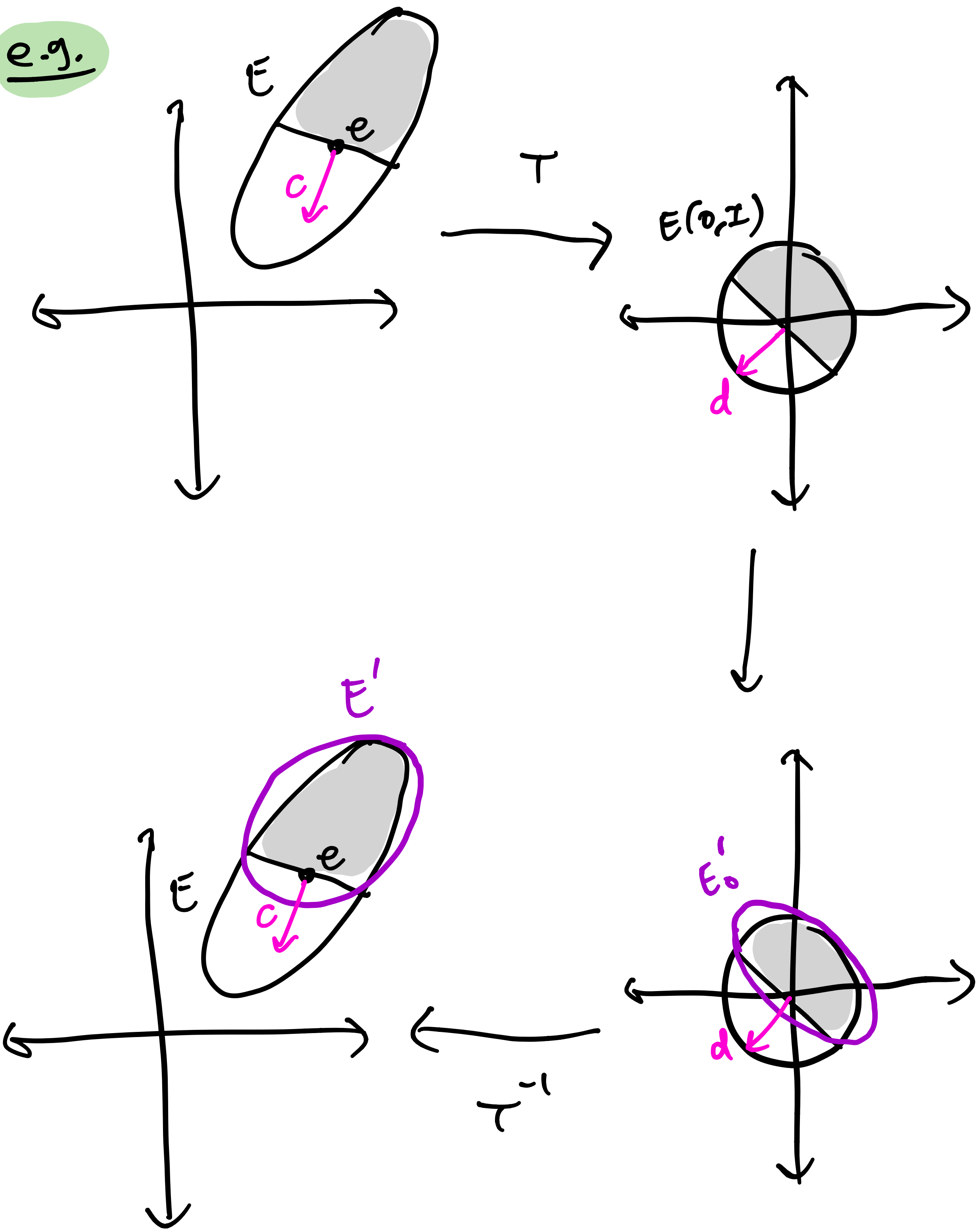
- What if E not unit ~~sphere~~ ball? Use affine transform T (preserves ratios of volumes) to turn E into unit ball.

$$\begin{tikzcd}
E \arrow[r, "T"] & E(0, I) \arrow[d] \\
E' & E'_0 \arrow[l, "T^{-1}"]
\end{tikzcd}$$

E' for ellipsoid E.

E'_0 for unit ball

e.g.



- Now

$$\frac{\mathrm{vol}\, E'}{\mathrm{vol}\, E} = \frac{\mathrm{vol}\, T^{-1} E_0'}{\mathrm{vol}\, E}$$

$$= \frac{\mathrm{vol}\, E_0'}{\mathrm{vol}\, \underset{\substack{\| \\ TE}}{E(0,I)}} \leq e^{-\frac{1}{2(n+1)}}$$

Completes proof of volume lemma. $\square$.

<u>How to compute E'?</u>

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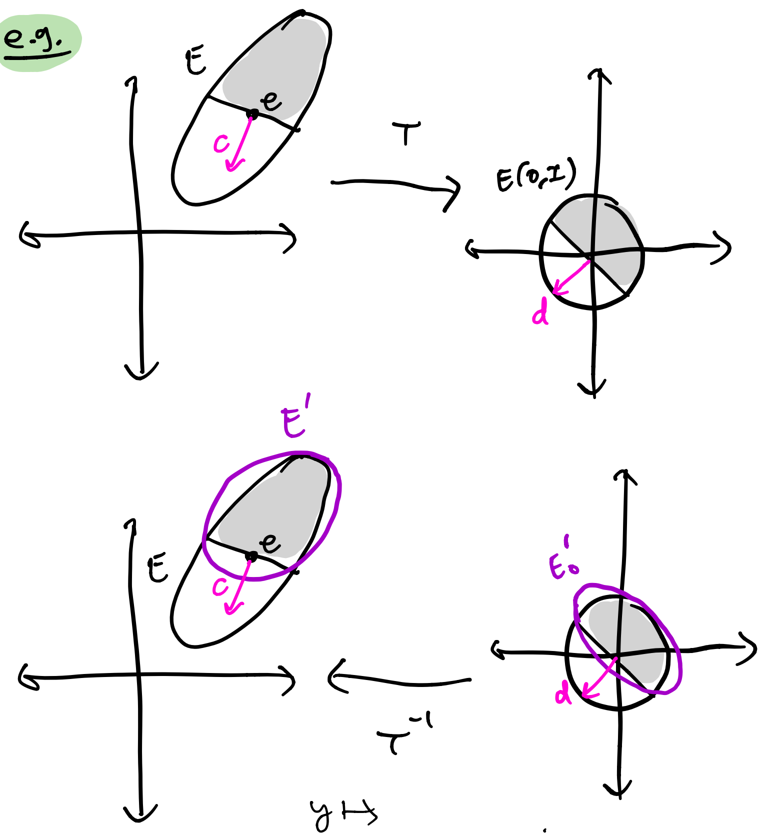
- Let's carefully compute

If $E = E(c, A)$, recall

$T : x \longmapsto y :=$

has

e.g.



- First find d. Under T,

$\{x : c^T x \le c^T e\} \xrightarrow[T]{} \{y: \quad\quad \}$

$= \{y: \quad\quad \} = \{y: \quad\quad \},$

for $d = \quad\quad = \quad\quad$

- Recall that

$E_0' = E(\quad , \quad\quad)$

- Let $b = \quad\quad = \quad\quad ;$

Applying T^{-1} to E_0' yields

$$E' = E(\quad , \quad).$$

$$= E(\quad , \quad).$$

Ellipsoid (Concretely):

- Initialize $E =$
- while $c \notin P$:
  - ▷ Let
  - ▷ Let $b =$

▷ set $e \leftarrow$

▷ set $A \leftarrow$

.

Analysis summary:

After k iterations,

$$\mathrm{vol}\, E \le e^{-\frac{k}{2(n+1)}} \mathrm{vol}\, E_0 \quad .$$

$\Rightarrow$

terminates in $\le$

$$2(n+1) \ln \frac{\mathrm{vol}\, E_0}{\mathrm{vol}\, P}$$

steps. (finds point in P).

Linear programming: